Hilbert series of algebras associated to directed graphs and order homology

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Abstract

We give a homological interpretation of the coefficients of the Hilbert series for an algebra associated with a directed graph and its dual algebra. This allows us to obtain necessary conditions for Koszulity of such algebras in terms of homological properties of the graphs. We use our results to construct algebras with a prescribed Hilbert series.

Key words: Hilbert series, directed graphs, order homology

Introduction

In [13,5,21,22,23,24,26] we introduced and studied certain associative non-commutative algebras $A(\Gamma)$ defined by layered graphs Γ (or ranked posets) and their generalizations. The algebras $A(\Gamma)$ are related to factorizations of polynomials with noncommutative coefficients and we called them *splitting algebras*. An important example of such algebras is the algebra Q_n defined by

the Boolean lattice of subsets of a finite set (see [14]) and related to the theory of noncommutative symmetric functions [7]. For *homogeneous* layered graphs [21] the algebras $A(\Gamma)$ are quadratic and one can construct their quadratic dual algebras $A(\Gamma)^!$.

It turns out that algebraic properties of $A(\Gamma)$ and $A(\Gamma)^!$ are closely related to homological properties of Γ. When a layered graph Γ is defined by a "good" regular cell complex the algebras $A(\Gamma)$ and $A(\Gamma)^!$ are Koszul if and only if all intermediate (order) cohomologies of Γ are trivial (see [3,24]).

Recall that if a quadratic algebra A is Koszul then it is *numerically* Koszul, i.e. the Hilbert series (or the graded dimension) $h(A, \tau)$ of A and the Hilbert series $h(A^!, \tau)$ of its dual algebra $A^!$ satisfy the identity

$$h(A, \tau)h(A^!, -\tau) = 1. \tag{1}$$

In this paper, motivated by the results from [3,24] and identity (1) we present a homological interpretation of coefficients of the Hilbert series for the algebras $A(\Gamma)$ and $A(\Gamma)^!$. In fact, instead of $A(\Gamma)^!$ we are working with a simpler algebra, the algebra $B(\Gamma)$, the graded algebra associated with a natural filtration on $A(\Gamma)^!$. The algebras $A(\Gamma)$, $A(\Gamma)^!$, $B(\Gamma)$ are Koszul if and only if at least one of these algebras is Koszul. Also, $h(A(\Gamma)^!, \tau) = h(B(\Gamma), \tau)$. Note that the algebras $B(\Gamma)$ are defined for any directed graph without any extra assumptions.

Our homological interpretation of the coefficients of the Hilbert series for the algebras $A(\Gamma)$ and $B(\Gamma)$ allows us to obtain conditions for their numerical Koszulity and also to construct algebras with prescribed Hilbert series defined, for example, by palindromic polynomials. Our formulas look particularly simple when certain posets associated with Γ are Cohen-Macaulay.

There are several ways to associate an algebra to a directed graph or a poset, and there are known connections between properties of such algebras and topological structures of graphs and posets (see [2]). The most famous example is the incidence algebra of a finite poset described in [29], Section 3.6. There is a certain resemblance between our algebra $A(\Gamma)^!$ and the incidence algebra defined by the graph Γ but our results are quite different.

The paper is organized in the following way. In Section 1 we recall basic facts about graphs, posets and their (co)homologies. Section 2 contains the definition of the algebras $A(\Gamma)$, $A(\Gamma)^!$ and $B(\Gamma)$. Our main theorem on homological description of the coefficients in the Hilbert polynomials for the algebras $B(\Gamma)$ and its corollaries are formulated in Section 3. Section 4 is devoted to numerical Koszulity of the algebras $A(\Gamma)$ and $B(\Gamma)$. Section 5 contains a number of examples including examples of algebras with Hilbert series equal to $P(-\tau)^{-1}$ where $P(\tau)$ is a palindromic polynomial. Calabi-Yau algebras also have Hilbert

series defined by palindromic polynomials (see [15]).

1 Partially ordered sets and directed graphs

1.1 Partially ordered sets and their homology

Let P be a partially ordered set (poset). Define an i-chain, i.e. a chain of length i, in P to be an $(i+1)$-tuple

$$\mathbf{x} = (x_0, ..., x_i)$$

of elements of P with

$$x_0 < x_1 < ... < x_i.$$

Let $Ch_i(P)$ denote the set of i-chains. The Möbius function, μ is defined by

$$\mu(x, y) = c_0 - c_1 + c_2 - \ldots$$

where c_i is the number of i-chains $\mathbf{x} = (x_0, ..., x_i)$ with $x = x_0, y = x_i$.

We say that $u \in P$ covers $v \in P$ (and write $v \prec u$) if $v < u$ and there is no element between u and v. If P is finite, denote by $l(P)$ the maximal length of a chain in P.

Denote by $\hat{P}$ the poset obtained from P by adding the minimal element $\hat{0}$ and the maximal element $\hat{1}$. We write $\mu(P)$ for $\mu(\hat{0}, \hat{1})$.

Suppose that P is a lattice, i.e. for any two elements $x, y \in P$ their least upper bound $x \vee y$ and the greatest lower bound $x \wedge y$ are defined. A lattice P of finite length is a lower semimodular lattice if for any two elements $x, y \in P$, if $x \vee y$ covers both x and y then x and y both cover $x \wedge y$.

Let F be a field and P be a finite poset. If $i \geq -1$, denote by $C_i(P; F)$ the free F-module on the set of i-chains $\mathbf{x}$ of P. The empty set is a (-1)-chain, so we identify $C_{-1}(P; F)$ with F. Set $C_n(P; F) = 0$ if $n < -1$.

If $\mathbf{x} = (x_0, ..., x_i)$ is an i-chain and $0 \leq l \leq i$, define $g^l(\mathbf{x})$ to be the $(i-1)$-chain

$$(x_0, ..., x_{l-1}, x_{l+1}, ..., x_i).$$

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Define a map $d_i : C_i(P; F) \to C_{i-1}(P; F)$ by linearly extending

$$d_i(\mathbf{x}) = \sum_{l=0}^{i} (-1)^l g^l(\mathbf{x}) \qquad (2)$$

when $i - 1$-chains exist, and setting $d_i = 0$ otherwise. It is easy to check that $d_i d_{i+1} = 0$, and one can define *reduced order homology groups* of P with coefficients in F:

$$\tilde{H}_i(P; F) = Ker \; d_i / Im \; d_{i+1}.$$

By construction, $\tilde{H}_i(P; F) = 0$ for $i < -1$ and $i > l(P)$. Also, $\tilde{H}_{-1}(P; F) = 0$ if and only if P is nonempty, and $\tilde{H}_0(P; F) = 0$ if and only if P is connected.

Denote by $C^i(P; F)$ the vector space dual to $C_i(P; F)$. For an i-chain $\mathbf{x}$ let $\mathbf{x}^*$ denote the element of $C^i(P; F)$ defined by

$$\mathbf{x}^*(\mathbf{y}) = \delta_{\mathbf{x}, \mathbf{y}}$$

for all i-chains $\mathbf{y}$. Define ∂^i to be the linear map dual to d_i. One can then define the cohomology groups

$$\tilde{H}^i(P; F) = Ker \; \partial^{i+1} / Im \; \partial^i.$$

The Möbius function is the Euler characteristic of reduced order homology:

$$\mu(P) = \sum_{i=-1}^{l(P)} (-1)^i \dim \; \tilde{H}_i(P; F).$$

Corresponding to a poset P there is a simplicial complex $\Delta(P)$. The vertices of $\Delta(P)$ are the elements of P and the i-faces of $\Delta(P)$ are the i-chains of P.

Recall, that a poset P is *Cohen - Macaulay* if for each open interval (x, y) in P all homology groups $\tilde{H}_i((x, y); F)$ are trivial for $i < \dim \; \Delta((x, y))$. Any lower semimodular lattice is Cohen-Macaulay.

From now on we will consider F to be fixed and write $\tilde{H}_i(P)$ for $\tilde{H}_i(P; F)$ and $\tilde{H}^i(P)$ for $\tilde{H}^i(P; F)$.

1.2 Layered graphs

Let $\Gamma = (V, E)$ be a directed graph (quiver) where V is the set of vertices and E is the set of edges. For any $e \in E$ denote by $t(e)$ the tail of e and by $h(e)$ the

head of e. A path π in Γ is a sequence of edges $\pi = (e_1, e_2, \ldots, e_k)$ such that $h(e_i) = t(e_{i+1})$ for $i = 1, 2, \ldots, k-1$. We call $t(e_1)$ the tail of π and denote it by $t(\pi)$ and we call $h(e_k)$ the head of π and denote it by $h(\pi)$.

Assume that $V = \coprod_{i=0}^{N} V_i$. We call N the *height* of the graph and we say that i is the *level* of v and write $|v| = i$ if $v \in V_i$. We call Γ a *layered* graph if $|t(e)| = |h(e)| + 1$ for any edge $e \in E$.

For any directed graph $\Gamma = (V, E)$ there is a corresponding partially ordered set. The elements of the poset are the vertices $v \in V$. We say that $u > v$ if and only if there is a directed path from u to v. By abuse of notation, we denote the poset by the same letter Γ. Therefore, we may talk about Möbius functions and (co)homologies of directed graphs by considering them as posets.

Obviously, for a layered graph u covers v if and only if there is an edge going from u to v.

We say that vertices v, v' of the same level $i > 0$ are connected by a *down-up* sequence if there exist vertices $v = v_0, v_1, v_2, \ldots, v_k = v' \in V_i$ and $w_1, w_2, \ldots, w_k \in V_{i-1}$ such that $w_j < v_{j-1}, v_j$ for $j = 1, 2, \ldots, k$. According to [21], the layered graph Γ is *uniform* if for any pair of edges $e, e' \in E$ with a common tail, $t(e) = t(e')$, their heads $h(e), h(e')$ are connected by a down-up sequence $v_0, \ldots, v_k$ such that $v_i < t(e)$, $i = 0, 1, \ldots, k$.

Let $\Gamma = (V, E)$ be a layered graph with $V = \coprod_{i=0}^{N} V_i$ and $V_i = \emptyset$ unless $r \leq i \leq r + L - 1$. Note that this implies that $Ch_L(\Gamma) = \emptyset$ and that if $\mathbf{x} = (x_0, \ldots, x_{L-1})$ is an $(L-1)$-chain then $x_i \in V_{r+i}$ for $0 \leq i \leq L - 1$.

For $0 \leq l \leq L - 1$ set

$$g^l(Ch_{L-1}(P)) = Ch_{L-2}^{(l)}(P)$$

and

$$g^l(C_{L-1}(P)) = C_{L-2}^{(l)}(P).$$

Let $C^{(l), L-2}(P)$ denote the set of all linear functions $f : C_{L-2}^{(l)}(P) \to F$. Now $Ch_{L-2}(P)$ is the disjoint union of the $Ch_{L-2}^{(l)}(P)$ and so

$$C^{L-2}(P) = \bigoplus_{l=0}^{L-1} C^{(l), L-2}(P)$$

where we extend $f \in C^{(l), L-2}(P)$ to a function on $C_{L-2}(P)$ by setting $f(C_{L-2}^{(m)}(P)) = 0$ for $m \neq l$.

Two families of layered graphs are particularly interesting.

Example 1.3.1 *Let* $\Gamma = (V, E)$ *where* $V = \coprod_{i=0}^{N} V_i$ *is a layered graph. We say that* Γ *is a complete layered graph if for every* $i, 1 \le i \le N$, *and every* $v \in V_i, w \in V_{i-1}$ *there is an edge from* v *to* w. *Clearly a complete layered graph is determined up to isomorphism by the integers* $|V_N|, ..., |V_1|, |V_0|$. *Let* $m_N, ..., m_0$ *be positive integers. Denote by* $\mathbf{C}[m_N, ..., m_0]$ *the complete layered graph with* $|V_i| = m_i$ *for* $0 \le i \le N$.

Proposition 1.3.2

$$\dim H^i(\mathbf{C}[m_N, ..., m_0]) = 0$$

if $-1 \le i \le N - 1$ *and*

$$\dim H^N(\mathbf{C}[m_N, ..., m_0]) = (m_N - 1)(m_{N-1} - 1)...(m_0 - 1).$$

Proof: The result is clear if $N = 0$ or -1. Hence assume $N > 1$. We first show that the partially ordered set V is lexicographically shellable (in the sense of Definition 2.2 of [2]). For each $i, 0 \le i \le N$ choose $x_i' \in V_i$. If $e \in E_i$, define $\lambda(e) = 1$ if $t(e) = x_i'$ and $\lambda(e) = N + 1 - i$ otherwise. Then if $[x, y]$ is an interval of V with $x \in V_i, y \in V_j$ we see that $x < x_{i+1}' < ... < x_{j-1}' < y$ is the unique rising unrefinable chain from x to y, so this labeling is an R-labeling. Furthermore if $x \prec z \le y$ and $z \ne x_{i+1}'$ we have $\lambda(x, x_{i+1}') = 1 < N + 1 - i = \lambda(x, z)$. Thus this labeling is an L-labeling and so V is lexicographically shellable. Then by Theorem 3.2 of [2] V is Cohen-Macaulay, giving the first statement.

Now
$$\dim C^i(\mathbf{C}[m_N, ..., m_0]) = \dim C_i(\mathbf{C}[m_N, ..., m_0]) =$$
$$|Ch_i(\mathbf{C}[m_N, ..., m_0])| = \sum_{|J|=i+1, J \subseteq \{0,...,N\}} \prod_{j \in J} m_j.$$

Thus, by the first part of the proposition and the Euler-Poincare principle,

$$\dim H^N(\mathbf{C}[m_N, ..., m_0]) = (-1)^N \sum_{i=-1}^{N} (-1)^i \dim H^i(\mathbf{C}[m_N, ..., m_0]) =$$

$$(-1)^N \sum_{i=-1}^{N} (-1)^i \dim C^i(\mathbf{C}[m_N, ..., m_0])) = (-1)^N \sum_{i=-1}^{N} (-1)^i \sum_{|J|=i+1} \prod_{j \in J} m_j =$$

$$= \sum_{J \subseteq \{0,...,N\}} (-1)^{N+1-|J|} \prod_{j \in J} m_j = \prod_{i=0}^{N} (m_i - 1).$$

Example 1.3.3 *Let Θ_N denote the Hasse graph of the (Boolean) lattice of all subsets of $\{1, ..., N\}$. Thus the vertices of Θ_N are the subsets of $\{1, ..., N\}$, the level of $Y \subseteq \{1, ..., N\}$ is $|Y|$, and there is an edge for $Y \subseteq \{1, ..., N\}$ to $Z \subseteq \{1, ..., N\}$ if and only if $Y \supseteq Z$ and $|Y| = |Z| + 1$.*

If Γ is any layered graph, $a \in V(\Gamma)$, and $i \leq |a|$, define $\Gamma_{a,i}$ to be the subgraph induced by the set of vertices

$$\{w \in V(\Gamma) | a > w \quad \text{and} \quad |a| - |w| \leq i - 1\}.$$

If Γ has unique maximal vertex v denote $\Gamma_{v,i}$ by Γ_i. Clearly $(\Theta_N)_{a,i}$ is isomorphic to $\Theta_{|a|,i}$.

We will need the following results about the order homology of Θ_N later.

Proposition 1.3.4 *If $2 \leq i \leq N$ and $1 \leq j \leq i - 3$ then*

$$H^j(\Theta_{N,i}) = (0).$$

Proof: Note that any edge e in $\Theta_{N,i}$ has tail S and head $S \setminus \{a\}$ for some $S \subseteq \{1, ..., N\}$ and some $a \in S$. Define $\lambda(e) = a$. Then λ is an L-labeling in the sense of Definition 2.2 of [2] and so, $\Theta_{N,i}$ is lexicographically shellable and hence (cf. [2]) $\Theta_{N,i}$ is Cohen-Macaulay, giving the result.

Proposition 1.3.5 *If $2 \leq i \leq N$ we have*

$$\dim \ H^{i-2}(\Theta_{N,i}) = \binom{N-1}{i-1}.$$

Proof: If $A \subseteq \{1, ..., N\}$, let A^c denote $\{1, ..., N\} \setminus A$. Let $N[i-1]$ denote the set of all ordered $(i-1)$-tuples $\mathbf{x} = (x_0, ..., x_{i-2})$ of distinct elements of $\{1, ..., N\}$. Note that Sym_{i-1}, the symmetric group on $\{0, ..., i-2\}$, acts on $N[i-1]$ by permuting subscripts. For $1 \leq l \leq i-2$ let σ_l denote the transposition interchanging $l-1$ and l.

If $\mathbf{x} = (x_0, ..., x_{i-2}) \in N[i-1]$ and $0 \le m \le i-2$ define

$$b_m(\mathbf{x}) = \{1, ..., N\} \setminus \{x_m, ..., x_{i-2}\}$$

and

$$b(\mathbf{x}) = (b_0(\mathbf{x}), ..., b_{i-2}(\mathbf{x})).$$

Then

$$b : N[i-1] \to Ch_{i-2}(\Theta_{N,i})$$

is a bijection and $\{b(\mathbf{x})^* | \mathbf{x} \in N[i-1]\}$ is a basis for $C^{i-2}(\Theta_{N,i})$.

Since b and g^l are both surjective, the composition

$$g^l b : N[i-1] \to Ch_{i-3}^{(l)}(\Theta_{N,i})$$

is surjective. Furthermore, if $1 \le l \le i-2$ and $\mathbf{x}, \mathbf{y} \in N[i-1]$ we see that

$$g^l b(\mathbf{x}) = g^l b(\mathbf{y})$$

if and only if

$$\mathbf{y} \in \{\mathbf{x}, \sigma_l \mathbf{x}\}.$$

Also, if $\mathbf{x} = (x_0, ..., x_{i-2}), \mathbf{y} = (y_0, ..., y_{i-2}) \in N[i-1]$, then

$$g^0 b(\mathbf{x}) = g^0 b(\mathbf{y})$$

if and only if

$$(x_1, ..., x_{i-2}) = (y_1, ..., y_{i-2}).$$

Now

$$d_{i-2}(b(\mathbf{y})) = \sum_{l=0}^{i-2} (-1)^l g^l b(\mathbf{y})$$

and so

$$\partial^{i-1}((g^l b(\mathbf{y}))^*) = (-1)^l \sum_{g^l b(\mathbf{x}) = g^l b(\mathbf{y})} b(\mathbf{x})^*.$$

Hence, if $1 \le l \le i-2$,

$$\partial^{i-2}(C^{(l),i-3}(\Theta_{N,i})) = span\{b(\mathbf{x})^* + b(\sigma_l \mathbf{x})^* | \mathbf{x} \in N(i-1)\} \subseteq C^{i-2}(\Theta_{N,i}).$$

Thus σ_l acts as the identity on $\partial^{i-2}(C^{(l),i-3}(\Theta_{N,i}))$ so setting

$$\Xi = \sum_{\sigma \in Sym_{i-1}} (sgn\ \sigma)\sigma$$

we see that

$$\Xi \partial^{i-2}(C^{(l),i-3}(\Theta_{N,i})) = (0)$$

for $1 \le l \le i-2$. Furthemore,

$$\partial^{i-2}(C^{(0),i-3}(\Theta_{N,i-1})) =$$

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$$\text{span } \{ \sum_{y_0 \in \{x_1,...,x_{i-2}\}^c} b(y_0, x_1, ..., x_{i-2})^* | \mathbf{x} = (x_0, ..., x_{i-2}) \in N[i-1]\}.$$

Consequently, if $x_0 = 1$, then

$$b(\mathbf{x})^* + \sum_{1 < y_0 \in \{x_1,...,x_{i-2}\}^c} b(y_0, x_1, ..., x_{i-2})^* \in \partial^{i-2}(C^{i-3}(\Theta_{N,i})).$$

Thus if

$$D = \text{span } \{b(\mathbf{x})^* | \mathbf{x} \in N[i-1], 1 < x_0 < x_1 < ... < x_{i-2}\}$$

we have

$$C^{i-2}(\Theta_{N,i}) = \partial^{i-2}(C^{i-3}(\Theta_{N,i})) + D.$$

Define a linear map

$$\Psi : C^{i-2}(\Theta_{N,i}) \to C^{i-2}(\Theta_{N,i})$$

by

$$\Psi(b(\mathbf{x})^*) = b(\mathbf{x})^* \text{ if } 1 \notin \{x_0, ..., x_{i-2}\}$$

and

$$\Psi(b(\mathbf{x})^*) = - \sum_{y_l \in \{x_0,...,x_{i-1}\}^c} b(x_0, ..., x_{l-1}, y_l, x_{l+1}, ..., x_{i-2})^* \text{ if } x_l = 1.$$

Then $\Psi\sigma = \sigma\Psi$ for all $\sigma \in Sym_{i-1}$. Thus

$$\Xi\Psi(\partial^{i-2}C^{(l),i-3}(\Theta_{N,i})) = (0)$$

for $1 \le l \le i - 2$.

Now if $\mathbf{x} \in N[i-1]$ and $1 \notin \{x_0, ..., x_{i-2}\}$ then $\Psi\partial^{i-2}(g^0 b(\mathbf{x})^*) = \Psi(b(1, x_1, ..., x_{i-2})^*) + \sum_{1 < y_0 \in \{x_1,...,x_{i-2}^c\}} b(y_0, x_1, ..., x_{i-2})^* = 0$. Also if $1 = x_l$ with $1 \le l \le i - 2$ we have

$$\Psi\partial^{i-2}((g^0 b(\mathbf{x}))^*) = - \sum_{y \ne z, y, z \in \{x_0,...,x_{i-2}\}^c} b(y, x_0, ..., x_{l-1}, z, x_{l+1}, ..., x_{i-2})^*.$$

Since this is invariant under the transposition that interchanges 0 and l, it is annihilated by Ξ. Thus

$$\Xi\Psi\partial^{i-2}(C^{(0),i-3}(\Theta_{N,i})) = (0)$$

and so

$$\Xi\Psi\partial^{i-2}(C^{i-3}(\Theta_{N,i})) = (0).$$

Now Ψ acts as the identity on D and Ξ is injective on D. Thus we have

$$C^{i-2}(\Theta_{N,i}) = \partial^{i-2}(C^{i-3}(\Theta_{N,i})) \oplus D.$$

Since

$$\dim\ D = \binom{N-1}{i-1},$$

the proof of the proposition is complete.

2 Algebras associated with layered graphs

2.1 The algebras $A(\Gamma)$

Following [13] we construct now an algebra $A(\Gamma)$ associated with a layered graph Γ. The algebra $A(\Gamma)$ is generated over the field F by generators $e \in E$ subject to the following relations. Let t be a formal parameter commuting with edges $e \in E$. Any two paths $\pi = (e_1, e_2, \ldots, e_k)$ and $\pi' = (f_1, f_2, \ldots, f_k)$ with the same tail and head define the relation

$$(t - e_1)(t - e_2)\ldots(t - e_k) = (t - f_1)(t - f_2)\ldots(t - f_k). \tag{3}$$

In fact, relation (3) is equivalent to k relations

$$e_1 + e_2 + \ldots + e_k = f_1 + f_2 + \ldots + f_k,$$

$$\sum_{i<j} e_i e_j = \sum_{i<j} f_i f_j,$$
$$\ldots$$
$$e_1 e_2 \ldots e_k = f_1 f_2 \ldots f_k.$$

We call $A(\Gamma)$ the *splitting* algebra associated with graph Γ. The terminology is justified by the following considerations. Assume that there are only one vertex $*$ of the minimal level 0 and only one vertex x of the maximal level N, and that for any edge e there exists a path $\theta = (e_1, e_2, \ldots, e_N)$ containing e from x to $*$. Set

$$P(t) = (t - e_1)(t - e_2)\ldots(t - e_N).$$

Then $P(t)$ is a polynomial over $A(\Gamma)$ and any path from the maximal to minimal vertex corresponds to a factorization of $P(t)$ into a product of linear factors.

Let $\Gamma = (V, E)$ be a layered graph. We assume that the layered graph Γ has exactly one minimal vertex so that for any vertex $v \in V_i$, $i > 0$ there is a path $\pi_v = (e_1, e_2, \ldots, e_i)$ from v to the minimal vertex. In this case the splitting algebra $A(\Gamma)$ is defined by a set of homogeneous relations of order 2 and higher.

It was proved in [21] that if the graph Γ is uniform then the splitting algebra $A(\Gamma)$ is quadratic, i.e. defined by relations of order 2.

Recall that for a quadratic algebra A over a field F there is a notion of the dual quadratic algebra $A^!$. To define $A^!$, denote by W the F-span of the generators of A and by $R \subset W \otimes W$ the linear space of relations of A. Denote by W^* the dual space of W and by $R^\perp$ the annihilator of R in $W^* \otimes W^*$. The algebra $A^!$ is the quadratic algebra defined by generators W^* and relations $R^\perp$. It is well-known (see, for example, [19]) that an algebra A is Koszul if and only if its dual algebra $A^!$ is. In this case their Hilbert series are connected by (1).

Assuming that a layered graph Γ is uniform one can describe the dual algebra $A(\Gamma)^!$ in terms of vertices and edges of the graph (see [23]). We describe now a slightly different algebra $B(\Gamma)$.

There is a natural filtration on $A(\Gamma)$ defined by the ranking function $|\cdot|$. The corresponding associated graded algebra is also quadratic. Its dual algebra $B(\Gamma)$ can be described in the following way (see [3]). Set $V_+ = \coprod_{i>0} V_i$. For any $v \in V_+$ let $S(v)$ be the set of all vertices $w \in V$ such that there is an edge going from v to w.

Theorem 2.2.1 *The algebra $B(\Gamma)$ is generated by vertices $v \in V_+$ subject to the relations:*

i) $u \cdot v = 0$ if there is no edge going from u to v;

ii) $v \cdot \sum_{w \in S(v)} w = 0$.

If the set of vertices V is finite, the algebra $B(\Gamma)$ is finite-dimensional. The algebras $B(\Gamma)$ were studied in [3]. According to the general theory $h(A(\Gamma)^!, \tau) = h(B(\Gamma), \tau)$, $A(\Gamma)$ is Koszul if and only if $A(\Gamma)^!$ is Koszul, and $A(\Gamma)^!$ is Koszul if and only if $B(\Gamma)$ is Koszul. Therefore, if either $A(\Gamma)$ or $B(\Gamma)$ is Koszul we have $h(B(\Gamma), \tau) = h(A(\Gamma), -\tau)^{-1}$.

3 Hilbert series for $B(\Gamma)$

3.1 Main theorem

Theorem 3.1.1 *Let $\Gamma = (V, E)$ be a uniform layered graph with $V = \coprod\limits_{i=0}^{N} V_i$. Then*

$$h(B(\Gamma), \tau) = 1 + \sum_{a \in V, |a| \geq i \geq 1} \dim\ (H^{i-2}(\Gamma_{a,i})) \tau^i.$$

We begin the proof with some preliminary remarks. We will write V^+ for the vector space with basis $\{v | v \in V_+\}$. Write

$$\overline{v} = \sum_{w \in S(v)} w.$$

By Theorem 2.2.1, $B(\Gamma)$ has presentation

$$B(\Gamma) = T(V^+)/(I + J)$$

where I is the ideal generated by

$$\{vw | v, w \in V_+, w \notin S(v)\}$$

and J is the ideal generated by

$$\{v\overline{v} | v \in V, 1 < |v|\}.$$

Let $v \in V_+$ and $n \in \mathbf{Z}$. If we set $T(V^+)_{v,n}$ equal to the span of all monomials $a_1...a_n$ with $a_1 = v$ and $a_2, ..., a_n \in V_+$, then we see that

$$T(V^+) = F + \oplus_{v \in V_+, 1 \leq n \leq |v|} T(V^+)_{v,n}.$$

Since the generators of I and J are homogeneous with respect to this decomposition and $T(V^+)_{v,n} \subseteq I$ for $n > |v|$, we see that, setting

$$I_{v,n} = I \cap T(V^+)_{v,n}$$

and

$$J_{v,n} = J \cap T(V^+)_{v,n}$$

we have

$$B(\Gamma) = \sum_{v \in V^+, 1 \leq n \leq |v|} B(\Gamma)_{v,n}$$

where

$$B(\Gamma)_{v,n} = T(V^+)_{v,n}/(I_{v,n} + J_{v,n}).$$

3.2 Proof of the theorem

Theorem 3.1.1 will follow from:

Proposition 3.2.1 $\dim(B(\Gamma)_{v,n}) = \dim(H^{n-2}(\Gamma_{v,n}))$.

Proof: If $j \geq 1$, $u \in V_{j+2}$, $w \in V_j$ define

$$\overline{uw} = \sum_{y \in S(u), w \in S(y)} y.$$

Recall that, for $0 \leq j \leq n-2$, $C_{n-3}^{(j)}(\Gamma_{v,n}) = g^j C_{n-2}(\Gamma_{v,n})$ and that $C^{(j),n-3}(\Gamma_{v,n})$ denotes the vector space of all functions

$$f : C_{n-3}^{(j)}(\Gamma_{v,n}) \to F.$$

Also write $Ch_{n-3}^{(j)}(\Gamma_{v,n}) = g^j Ch_{n-2}(\Gamma_{v,n})$.

Clearly

$$Ch_{n-3}(\Gamma_{v,n}) = \bigcup_{j=0}^{n-2} Ch_{n-3}^{(j)}(\Gamma_{v,n})$$

and so

$$C^{n-3}(\Gamma_{v,n}) = \bigoplus_{j=0}^{n-2} C^{(j),n-3}(\Gamma_{v,n})$$

where we extend $f \in C^{(j),n-3}(\Gamma_{v,n})$ to $Ch_{n-3}(\Gamma_{v,n})$ by setting $f(Ch_{n-3}^{(l)}(\Gamma_{v,n})) = 0$ for $j \neq l$.

Let $X \subseteq T(V^+)_{v,n}$ denote the span of all monomials $vb_{n-2}...b_0$ where $(b_0, ..., b_{n-2}) \in Ch_{n-2}(\Gamma_{v,n})$. Also, let

$$X_1 = span \ \{v\overline{v}b_{n-3}...b_0 | (b_0, ..., b_{n-3}) \in Ch_{n-3}^{(n-2)}(\Gamma_{v,n})\},$$

$$X_j = span \ \{vb_{n-3}...b_{n-1-j}(\overline{b_{n-1-j}b_{n-2-j}})b_{n-2-j}...b_0 | (b_0, ..., b_{n-3}) \in Ch_{n-3}^{(n-1-j)}(\Gamma_{v,n})\}$$

if $2 \leq j \leq n - 2$, and

$$X_{n-1} = span \ \{vb_{n-3}...b_0\overline{b_0} | (b_0, ..., b_{n-3}) \in Ch_{n-3}^{(0)}(\Gamma_{v,n})\}.$$

Then we have
$$X_1, ..., X_{n-1} \subseteq X,$$

$$X \cap I_{v,n} = (0),$$
$$X + I_{v,n} = T(V^+)_{v,n}$$

and

$$J_{v,n} + I_{v,n} = I_{v,n} + \sum_{j=1}^{n-1} X_j.$$

Therefore,

$$B(\Gamma)_{v,n} = T(V^+)_{v,n}/(I_{v,n} + J_{v,n}) \cong$$
$$(X + I_{v,n})/(I_{v,n} + J_{v,n}) \cong$$
$$(X + (I_{v,n} + \sum_{j=1}^{n-1} X_j))/(I_{v,n} + \sum_{j=1}^{n-1} X_j) \cong$$
$$X/(X \cap (I_{v,n} + \sum_{j=1}^{n-1} X_j)) = X/(\sum_{j=1}^{n-1} X_j).$$

Define
$$\psi : C^{n-2}(\Gamma_{v,n}) \to T(V^+)_{v,n}$$

by
$$\psi : f \mapsto \sum_{(b_0,...,b_{n-2}) \in Ch_{n-2}(\Gamma_{v,n})} f((b_0, ..., b_{n-2})) v b_{n-2}...b_0.$$

Also, for $0 \le j \le n - 2$, define

$$\psi_j : C^{(j),n-3}(\Gamma_{v,n}) \to T(V^+)_{v,n}$$

by
$$\psi_j : f \mapsto \sum_{(b_0,...,b_{n-2}) \in Ch_{n-2}(\Gamma_{v,n})} f(b_0, ..., \hat{b}_j, ..., b_{n-2}) v b_{n-2}...b_0.$$

Then ψ is an isomorphism of $C^{n-2}(\Gamma_{v,n})$ onto X, and, for $0 \le j \le n - 2$, ψ_j is an isomorphism of $C^{(j),n-3}(\Gamma_{v,n})$ onto X_{n-1-j}.

For $f \in C^{n-3}(\Gamma_{v,n})$ write $f = \sum_{j=0}^{n-2} f_j$ with $f_j \in C^{(j),n-3}(\Gamma_{v,n})$.

Observe that

$$\psi \partial^{n-2} f = \sum_{j=0}^{n-2} (-1)^j \psi_j f_j.$$

For $0 \le j \le n - 2$, define

$$C^{[j],n-3}(\Gamma_{v,n}) = \sum_{l=0}^{j} C^{(l),n-3}(\Gamma_{v,n}).$$

Note that
$$C^{[0],n-3}(\Gamma_{v,n}) = C^{(0),n-3}(\Gamma_{v,n})$$
and so
$$C^{[0],n-3}(\Gamma_{v,n}) \cap ker\partial^{n-2} = (0).$$
Also note that
$$C^{[n-2],n-3}(\Gamma_{v,n}) = \sum_{l=0}^{n-2} C^{(l),n-3}(\Gamma_{v,n}) = C^{n-3}(\Gamma_{v,n}).$$

Then if $f \in C^{[j],n-3}(\Gamma_{v,n}) \cap ker\partial^{n-2}$ we have

$$0 = \sum_{l=0}^{j}(-1)^l \psi_l f_l$$

and so

$$\psi_j f_j = \sum_{l=0}^{j-1}(-1)^{l-j-1}\psi_l f_l \in X_{n-1-j} \cap (X_{n-j} + ... + X_{n-1}).$$

Furthermore, since each ψ_l is surjective, ψ_j maps $C^{[j],n-3}(\Gamma_{v,n})$ onto $X_{n-1-j} \cap (X_{n-j} + ... + X_{n-1})$.

The kernel of the restriction of ψ_j to

$$C^{[j],n-3}(\Gamma_{v,n}) \cap ker\partial^{n-2}$$

is clearly

$$C^{[j-1],n-3}(\Gamma_{v,n}) \cap ker\partial^{n-2}.$$

Thus

$$\dim(X_{n-1-j} \cap (X_{n-j} + ... + X_{n-1})) = \\ \dim(C^{[j],n-3}(\Gamma_{v,n}) \cap ker\partial^{n-2})- \\ \dim(C^{[j-1],n-3}(\Gamma_{v,n}) \cap ker\partial^{n-2}).$$

Now we have seen that

$$B(\Gamma)_{v,n} \cong X/(\sum_{j=1}^{n-1} X_j).$$

Since

$$\dim(\sum_{j=1}^{n-1} X_j) = \sum_{j=1}^{n-1}\dim(X_j) - \sum_{j=1}^{n-2}\dim(X_j \cap (X_{j+1} + ... + X_{n-1}))$$

we have

$$\dim(B(\Gamma)_{v,n}) = \dim(C^{n-2}(\Gamma_{v,n})) - \sum_{j=0}^{n-2}\dim(C^{(j),n-3}(\Gamma_{v,n}))+$$

$$\dim(C^{[0],n-3}(\Gamma_{v,n}) \cap ker\partial^{n-2}) - \dim(C^{[n-2],n-3}(\Gamma_{v,n}) \cap ker\partial^{n-2}) =$$

$$\dim(C^{n-2}(\Gamma_{v,n})) - \dim(C^{n-3}(\Gamma_{v,n})) + \dim(ker\partial^{n-2}) =$$

$$\dim(C^{n-2}(\Gamma_{v,n})) - \dim(\partial^{n-2}(C^{n-3}(\Gamma_{v,n}))).$$

Since $\partial^{n-1} = 0$ this shows that

$$\dim(B(\Gamma)_{v,n}) = \dim(H^{n-2}(\Gamma_{v,n})),$$

proving the proposition.

We may give an explicit description of the coefficients of τ^i in the Hilbert series of $B(\Gamma)$ for $0 \le i \le 3$. For $x \in \coprod_{i \ge 3} V_i$ write $\mathcal{E}(x,3)$ for the number of edges in the graph $\Gamma_{x,3}$ and $\mathcal{V}(x,3)$ for the number of vertices in the graph $\Gamma_{x,3}$.

Corollary 3.2.2

$$h(B(\Gamma), \tau) \equiv 1 + |V_+|\tau + (\sum_{i=2}^{n} |E_i| - \sum_{i=2}^{n} |V_i|)\tau^2 +$$

$$\sum_{|x| \ge 3} (\mathcal{E}(x,3) - \mathcal{V}(x,3) + 1)\tau^3 \ mod \ (\tau^4).$$

Proof: For any vertex a and any vertex b of $\Gamma_{a,1}$ we have $a > b$ and $|a| - |b| \le 0$. Thus the set of vertices of $\Gamma_{a,1}$ is empty and so we have $\dim H^{-1}(\Gamma_{a,1}) = 1$ for all $a \in V_+$. Since $\Gamma_{a,2}$ is the graph induced by $S(a)$ we have $\dim H^0(\Gamma_{a,2}) = |S(a)| - 1$ for any $a \in V_+$. Now $\sum_{a \in V_j} |S(a)| = |E_j|$ and so

$$\sum_{a \in V, |a| \ge 2} \dim (H^0(\Gamma_{a,2})) = \sum_{i=2}^{n} |E_i| - \sum_{i=2}^{n} |V_i|.$$

Let $x \in V_i, i \ge 3$, u be a vertex of $\Gamma_{x,3}$ of level $i - 1$ and v be a vertex of $\Gamma_{x,3}$ of level $i - 2$. Consider $(u)^*$ and $(v)^* \in C^0(\Gamma_{x,3})$. We have $\partial^1(u)^* = \sum_{w \in S(u)}(w, u)^*$ and $\partial^1(v^*) = -\sum_{w \in S(x), v \in S(w)}(v, w)^*$. Now consider the 0-chain $Y = \sum_{y \in V_1 \cup V_2} a_y(y)^*$. The coefficient of $(v, u)^*$ in $\partial^1 Y$ is 0 unless there is an edge from u to v and, if there is such an edge e, it is $-a_{t(e)} + a_{h(e)}$. Thus $Y \in ker \ \partial^1$ if and only if a_y is constant on each connected component of $\Gamma_{x,3}$. Since Γ is uniform, $\Gamma_{x,3}$ is connected. It follows that $\dim H^1(\Gamma_{x,3}) = \dim C^1(\Gamma_{x,3}) - \dim C^0(\Gamma_{x,3}) + 1 = \mathcal{E}(x,3) - \mathcal{V}(x,3) + 1$.

4 $A(\Gamma)$ and numerical Koszulity

4.1 Hilbert series of $A(\Gamma)$

Theorem 4.1.1 *Let $\Gamma = (V, E)$ be a uniform layered graph with $V = \coprod_{i=0}^{N} V_i$.*
Then:

$$h(A(\Gamma), \tau)^{-1} = 1 - \sum_{a \in V, |a| \geq i, 0 \leq s \leq i-1} (-1)^s \ \dim(H^{s-1}(\Gamma_{a,i})) \tau^i.$$

Proof: We will use a result from [24]. In Lemma 1.3 of that paper we show that if Γ is a uniform layered graph and if, for integers g, h, we define

$$s_{g,h} = \sum_{v_1 > ... > v_l > *, |v_1| = g, |v_l| = h} (-1)^l,$$

then

$$h(A(\Gamma), \tau)^{-1} = 1 + \sum_{i \geq 1} \Big(\sum_{g,h \in \mathbf{Z}, g \geq i \geq g-h+1} s_{g,h} \Big) \tau^i.$$

Now, for a fixed i, and for $a \in V$ with $|a| = g$, the chain

$$a = v_1 > ... > v_l > *$$

with $|v_l| = h$ and $|v_1| - |v_l| \leq i-1$ occurs in the index set for the sum defining $s_{g,h}$ if and only if $(v_l, ..., v_2) \in Ch_{l-2}(\Gamma_{a,i})$. Thus

$$\sum_{g \geq i \geq g-h+1} s_{g,h} = \sum_{|a|=g \geq i} \sum_{l=1}^{i} (-1)^l |Ch_{l-2}(\Gamma_{a,i})| =$$

$$\sum_{|a|=g \geq i} \sum_{l=1}^{i} (-1)^l \ \dim \ C^{l-2}(\Gamma_{a,i}).$$

Applying the Euler-Poincaré principle and setting $s = l - 1$ we obtain

$$- \sum_{|a|=g \geq i} \sum_{s=0}^{i-1} (-1)^s \ \dim \ H^{s-1}(\Gamma_{a,i}),$$

proving the result.

4.2 Numerical Koszulity

A version of the following theorem was announced in [20].

Theorem 4.2.1 *Let $\Gamma = (V, E)$ be a uniform layered graph with $V = \coprod_{i=0}^{N} V_i$. Assume that all minimal vertices of Γ are contained in V_0. Then $A(\Gamma)$ is numerically Koszul if and only if*

$$0 = \sum_{a \in V, |a| \geq i, 0 \leq s \leq i-2} (-1)^s \dim(H^{s-1}(\Gamma_{a,i}))$$

for all $i, 3 \leq i \leq N$.

Proof: We see from Theorems 3.1.1 and 4.1.1 that $A(\Gamma)$ is numerically Koszul if and only if

$$0 = \sum_{i \geq 1} \sum_{a \in V, |a| \geq i, 0 \leq s \leq i-2} (-1)^s \dim(H^{s-1}(\Gamma_{a,i})) \tau^i.$$

Now the sum giving the coefficient of τ is empty. The sum giving the coefficient of τ^2 is

$$\sum_{a \in V, |a| \geq 2} \dim(H^{-1}(\Gamma_{a,2})).$$

Since $\Gamma_{a,2}$ is the graph induced by the nonempty set of vertices $S(a)$, $\dim(H^{-1}(\Gamma_{a,2})) = 0$. Thus the coefficients of τ and τ^2 are always 0 so we have the result.

5 Examples

5.1 Complete layered graphs and Boolean graphs

We begin with two corollaries using the results of Section 1.3.

Example 5.1.1 *The algebras $A(\mathbf{C}[m_N, ..., m_1, 1])$ and $B(\mathbf{C}[m_N, ..., m_1, 1])$ are numerically Koszul and*

$$h(B(\mathbf{C}[m_n, ..., m_1, 1]), \tau) = 1 + \sum_{k=1}^{N} \sum_{l=k}^{N} m_l(m_{l-1} - 1)...(m_{l-k+1} - 1)\tau^k.$$

Proof: Note that if a is a vertex of $\mathbf{C}[m_N, ..., m_1, 1]$ of level j, then

$$\mathbf{C}[m_N, ..., m_1, 1]_{a,i} \cong \mathbf{C}[m_{j-1}, ..., m_{j-i+1}].$$

Then Proposition 1.3.2 and Theorems 3.1.1 and 4.1.1 give the result.

In fact, the algebras $A(\mathbf{C}[m_N, ..., m_1, 1])$ and $B(\mathbf{C}[m_N, ..., m_1, 1])$ are known to be Koszul [24] and the Hilbert series were computed in [22].

Example 5.1.2 *The algebras $A(\Theta_N)$ and $B(\Theta_N)$ are numerically Koszul and*

$$h(B(\Theta_N), \tau) = 1 + \sum_{i=1}^{N} \sum_{k=i}^{N} \binom{N}{k} \binom{N-1}{i-1} \tau^i.$$

Proof: This follows from Theorems 3.1.1 and 4.1.1 together with the computations of Propositions 1.3.4 and 1.3.5.

In fact, the algebras $A(\Theta_N)$ and $B(\Theta_N)$ are known to be Koszul [24] and the Hilbert series were computed in [4] and [28].

5.2 Algebras with prescribed Hilbert series

We may use the results of this section to determine the Hilbert series of $B(\Gamma)$ for graphs $\Gamma = (V, E)$ where $V = \coprod_{i=0}^{3} V_i$.

Example 5.2.1 *Let $\Gamma = (V, E)$ be a uniform layered graph with all minimal vertices contained in V_0. If $V = \coprod_{i=0}^{3} V_i$, and $V_3 = \{a\}$ and $a > b$ for any $b \in V_2$, then $A(\Gamma)$ and $B(\Gamma)$ are Koszul and*

$$h(B(\Gamma), \tau) = 1 + |V_+|\tau + (|E_2| - 1)\tau^2 + (|E_2| - |V_1| - |V_2| + 1)\tau^3.$$

Proof: Since $\Gamma_{a,3} \neq \emptyset$, $H^{-1}(\Gamma_{a,3}) = 0$. Then by Theorem 4.2.1, $B(\Gamma)$ is numerically Koszul if and only if $H^0(\Gamma_{a,3}) = (0)$. Now $\dim \partial^0(C^{-1}(\Gamma_{a,3})) = 1$ and, by the argument of the proof of Corollary 3.2.2, $\dim \ker(\partial^1(C^0(\Gamma_{a,3}))) = 1$. Thus $H^0(\Gamma_{a,3}) = (0)$ and $B(\Gamma)$ is numerically Koszul. By [24], since $V = \coprod_{i=0}^{3} V_i$, the numerical Koszulity of $B(\Gamma)$ implies Koszulity. The expression for the Hilbert series follows from Corollary 3.2.2.

Example 5.2.2 *Let Γ be a graph satisfying the conditions of Example 5.2.1.*

Set $r = |V_+|$ and $s = |E_2| - 1$. Then $r \geq 3$, $r - 3 \leq s \leq -1 + (r-1)^2/4$ and the Hilbert series of $B(\Gamma)$ is

$$h(B(\Gamma), \tau) = 1 + r\tau + s\tau^2 + (s - r + 3)\tau^3.$$

Conversely, if $r, s \in \mathbf{Z}$ satisfy the above conditions, then there is a graph satisfying the conditions of Example 5.2.1 with $r = |V_+|$ and $s = |E_2| - 1$.

Proof: Since $V_1, V_2, V_3 \neq \emptyset$ we have $r \geq 3$. Since Γ is uniform, $\Gamma_{a,3}$ must be connected and hence has at least $|V_1| + |V_2| - 1$ edges. Of course, $\Gamma_{a,3}$ can have at most $|V_1||V_2|$ edges. Thus $s = |E_2| - 1 \geq |V_1| + |V_2| - 2 \geq |V_+| - 3 = r - 3$. Also $s + 1 \leq |V_1|(r - 1 - |V_1|)$. Since the maximum value of $x(r - 1 - x)$ is $(r-1)^2/4$ we have the remaining inequality. To prove the existence of such a graph, let $\Gamma' = (V', E')$ where $V_3' = \{a\}$, $V_2' = \{b_1, ..., b_{[(r-1)/2]}\}$, $V_1' = c_1, ..., c_{r-1-[(r-1)/2]}$ and where there are edges from a to every b_i, from b_1 to every c_i, and from every b_i to c_1. Then the graph $\Gamma'_{a,3}$ is connected, so it satisfies the required conditions with $s = r - 3$. By adding additional edges connecting vertices in V_1 and V_2 we may attain examples with t edges where $r - 3 \leq t \leq (r-1)^2/4$ if r is odd and $r - 3 \leq t \leq r(r-2)/4$ if r is even.

Example 5.2.3 *Let $r \geq 9$. Then there is a uniform layered graph $\Gamma = (V, E)$ with $V = \coprod_{i=0}^{3} V_i$, $V_0 = \{*\}$, and $|V_3| = 1$ such that $A(\Gamma)$ is a Koszul algebra with Hilbert series*

$$(1 - r\tau + r\tau^2 - \tau^3)^{-1}.$$

Proof: For example, let

$$V_3 = \{a\},$$

$$V_2 = \{b_1, b_2, e_1, e_2, \ldots, e_{r-7}\},$$

$$V_1 = \{c_1, c_2, d_1, d_2\},$$

$$E_3 = \{(a, b_1), (a, b_2)\},$$

$$E_2 = \{(b_i, c_j), (e_i, c_j), (e_i, d_j) \mid i, j = 1, 2\} \cup \{(e_i, d_j) \mid 3 \leq i \leq r - 7, j = 1, 2\},$$

$$E_1 = \{(y, *) \mid y \in V_1\}$$
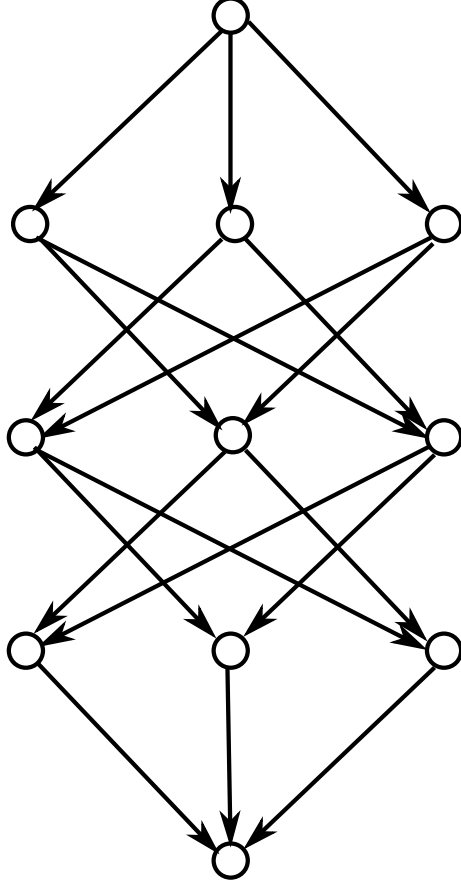
and apply Corollary 3.2.2.

Fig. 1. Cassidy-Shelton graph

5.3   Algebras which are not numerically Koszul

Example 5.3.1 *Figure 1 shows a uniform layered graph* $\Gamma = (V, E)$ *(due to Cassidy and Shelton) with* $V = \coprod_{i=0}^{4} V_i$ *such that* $A(\Gamma)$ *is not Koszul and, in fact, is not numerically Koszul. This example may be described as follows:*

$$V_4 = \{a\}, V_3 = \{b_1, b_2, b_3\}, V_2 = \{c_1, c_2, c_3\}, V_1 = \{d_1, d_2, d_3\}, V_0 = \{*\};$$

$$E_4 = \{(a, b_i)|i = 1, 2, 3\}, E_3 = \{(b_i, c_j)|i \neq j\}, E_2 = \{(c_i, d_j)|i \neq j\}, E_1 = \{(d_i, *)|i = 1, 2, 3\}.$$

Proof: We observe that

$$\dim H^{-1}(\Gamma_{a,4}) - \dim H^0(\Gamma_{a,4}) + \dim H^1(\Gamma_{a,4}) =$$

$$\dim C^{-1}(\Gamma_{a,4}) - \dim C^0(\Gamma_{a,4}) + \dim C^1(\Gamma_{a,4}) - \dim C^2(\Gamma_{a,4}) + \dim H^2(\Gamma_{a,4}) \geq$$
$$\dim C^{-1}(\Gamma_{a,4}) - \dim C^0(\Gamma_{a,4}) + \dim C^1(\Gamma_{a,4}) - \dim C^2(\Gamma_{a,4}) =$$
$$1 - 9 + 21 - 12 = 1$$

so Theorem 4.2.1 shows that $B(\Gamma)$ is not numerically Koszul. A similar calculation shows that the graph obtained from the Cassidy-Shelton example by deleting the edge (b_3, c_2) also fails to be numerically Koszul.

References

[1]   Björner A., Topological Methods, In: Handbook of Combinatorics, MIT Press, Cambridge, MA, 1995, 1819–1872

[2]   Björner A., Shellable and Cohen-Macaulay partially ordered sets, Trans. Amer. Math. Soc., 1980, **260**, 159–183

[3]   Cassidy T., Phan C., Shelton B., Noncommutative Koszul algebras from combinatorial topology, J. Reine Angew. Math. 2010, **646**, 45–63

[4]   Gelfand I., Gelfand S., Retakh V., Serconek S., Wilson R., Hilbert series of quadratic algebras associated with decompositions of noncommutative polynomials, J. Algebra, 2002, **254**, 279–299

[5]   Gelfand I., Gelfand S., Retakh V., Wilson R., Quasideterminants, Advances in Math., 2005, **193**, 56–141

[6]   Gelfand I., Gelfand S., Retakh V., Wilson R., Factorizations of polynomials over noncommutative algebras and sufficient sets of edges in directed graphs, Lett. Math. Physics, 2005, **74**, 153–167

[7]   Gelfand I., Krob D., Lascoux A., Leclerc B., Retakh V., Thibon J.-Y., Noncommutative symmetric functions, Advances in Math., 1995, **112**, 218–348

[8]   Gelfand I., Retakh V., Determinants of matrices over moncommutative rings, Funct. Anal. Appl., 1991, **25**, 91–102

[9]   Gelfand I., Retakh V., A theory of noncommutative determinants and characteristic functions of graphs, Funct. Anal. Appl., 1992, **26**, no 4, 1–20

[10]   Gelfand I., Retakh V., A theory of noncommutative determinants and characteristic functions of graphs. I, Publ. LACIM, UQAM, Montreal, 1993, 1–26

[11]   Gelfand I., Retakh V., Noncommutative Vieta theorem and symmetric functions, In: Gelfand Mathematical Seminars 1993–95, Birkhäuser, Boston, 1996, 93–100

[12]   Gelfand I., Retakh V., Quasideterminants, I, Selecta Math., 1997, **3**, 517–546

[13]   Gelfand I., Retakh V., Serconek S., Wilson R., On a class of algebras associated to directed graphs, Selecta Math., 2005, **11**, 281–295

[14]   Gelfand I., Retakh V., Wilson R., Quadratic-linear algebras associated with decompositions of noncommutative polynomials and differential polynomials, Selecta Math., 2001, **7**, 493–523

[15]   V. Ginzburg, Algebras Calabi-Yau, arXiv: math/0612139

[16]   Osofsky B., Quasideterminants and right roots of polynomials over division rings, In: Algebras, Rings and Their Representations, World Sci. Publ., Singapore, 2006, 241–263

[17]   Osofsky B., Noncommutative linear algebra, Contemp. Math., 2006, **419**, 231–254

[18]   Piontkovski D., Algebras associated to pseudo-roots of noncommutative polynomials are Koszul, Intern. J. Algebra Comput., 2005, **15**, 643–648

[19]   Polischuk A., Positselski L., Quadratic Algebras, Amer. Math. Soc., Providence, RI, 2005

[20]   Retakh V., From factorizations of noncommutative polynomials to combinatorial topology, Central European J. Math., 2010, **8**, 235–243

[21]   Retakh V., Serconek S., Wilson R., On a class of Koszul algebras associated to directed graphs, J. Algebra, 2006, **304**, 1114–1129

[22]   Retakh V., Serconek S., Wilson R., Hilbert series of algebras associated to directed graphs, J. of Algebra, 2007, **312**, 142–151

[23]   Retakh V., Serconek S., Wilson R., Construction of some algebras associated to directed graphs and related to factorizations of noncommutative polynomials, Contemp. Math., 2007, **442**, 201–219

[24]   Retakh V., Serconek S., Wilson R., Koszulity of splitting algebras associated with cell complexes, J. Algebra, 2010, **323**, 983-999

[25]   Retakh V., Wilson R., Advanced course on quasideterminants and universal localization, CRM, Barcelona, 2007

[26]   Retakh V., Wilson R., Algebras associated to directed acyclic graphs, Adv. Appl. Math., 2009, **42**, 42–59

[27]   Sadofsky H., Shelton B., The Koszul property as a topological invariant and measure of singularities, preprint avaluable at arXiv:0911.2541

[28]   Serconek S., Wilson R., Quadratic algebras associated with decompositions of noncommutative polynomials are Koszul algebras, J. Algebra, 2004, **278**, 473–493

[29]   Stanley R., Enumerative Combinatorics, Vol 1, Cambridge Univ. Press, 1997